A List of Laplace and Inverse Laplace Transforms Related to Fractional Order Calculus

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The following collection is based on [1, 2, 3].

1 Some special function of the Mittag-Leffler type

A two parameters function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha k + \beta)}, \qquad (\alpha > 0,\ \beta > 0),$$

and $E^{(k)}_{\alpha,\beta}(z)$ is its k-th derivative, and

$$\varepsilon_k(t, \lambda; \alpha, \beta) = t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\lambda t^{\alpha})$$

is the function introduced by Podlubny in [3] for solving of the fractional differential equations.

References

[1] A. D. Poularikas, *The handbook of formulas and table for signal processing*, The Electrical Engineering Handbook Series. CRC Press LLC and IEEE Press, New York, 1999.

[2] P. A. McCollum and B. F. Brown, *Laplace Tranform Tables and Theorems*, Holt Rinehart and Winston, New York, 1965.

[3] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

$F(s)$	$f(t)$
$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\frac{1}{s\sqrt{s}}$	$2\sqrt{\frac{t}{\pi}}$
$\frac{1}{s^n\sqrt{s}}$, $(n=1,2,\cdots)$	$\frac{2^n t^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}$
$\frac{s}{(s-a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$
$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\mathrm{erfc}(a\sqrt{t})$
$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\mathrm{erf}(a\sqrt{t})$
$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\tau^2}\mathrm{d}\tau$
$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\mathrm{erf}(a\sqrt{t})$
$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\tau^2}\mathrm{d}\tau$
$\frac{b^2-a^2}{(s-a^2)(\sqrt{s}+b)}$	$e^{a^2t}[b-a\mathrm{erf}(a\sqrt{t})]-be^{b^2t}\mathrm{erfc}(b\sqrt{t})$
$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2t}\mathrm{erfc}(a\sqrt{t})$
$\frac{1}{\sqrt{s+b}(s+a)}$	$\frac{1}{\sqrt{b-a}}e^{-at}\mathrm{erf}(\sqrt{b-a}\sqrt{t})$
$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2t}[\frac{b}{a}\mathrm{erf}(a\sqrt{t})-1]+e^{b^2t}\mathrm{erfc}(b\sqrt{t})$
$\frac{(1-s)^n}{s^{n+(1/2)}}$	$\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(\sqrt{t})$ where $H_n(x)=e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$, Hermite polynomial.
$\frac{(1-s)^n}{s^{n+(3/2)}}$	$-\frac{n!}{(2n+1)!\sqrt{\pi}}H_{2n+1}(\sqrt{t})$
$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s}}$	$ae^{-at}[I_1(at)+I_0(at)]$ where $I_n(x)=j^{-n}J_n(jt)$ and J_n is Bessel's function of the first kind.
$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t}I_0(\frac{a-b}{2}t)$
$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}$, $(k\geq 0)$	$\sqrt{\pi}(\frac{t}{a-b})^{k-(1/2)}e^{-\frac{1}{2}(a+b)t}I_{k-(1/2)}(\frac{a-b}{2}t)$
$\frac{1}{(s+a)^{1/2}(s+b)^{3/2}}$	$te^{-\frac{1}{2}(a+b)t}[I_0(\frac{a-b}{2}t)+I_1(\frac{a-b}{2}t)]$
$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t}e^{-at}I_1(at)$
$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}$, $(k>0)$	$\frac{k}{t}e^{-\frac{1}{2}(a+b)t}I_k(\frac{a-b}{2}t)$
$\frac{1}{\sqrt{s}\sqrt{s+a}(\sqrt{s+a}+\sqrt{s})^{2\nu}}$, $(k>0)$	$\frac{1}{a^\nu}e^{-\frac{1}{2}at}I_\nu(\frac{a}{2}t)$
$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
$\frac{1}{\sqrt{s^2-a^2}}$	$I_0(at)$, modified Bessel function of the first kind, zeroth order.
$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}$, $(\nu>-1)$	$a^\nu J_\nu(at)$
$\frac{1}{(\sqrt{s^2+a^2})^k}$, $(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}(\frac{t}{2a})^{k-(1/2)}J_{k-(1/2)}(at)$
$(\sqrt{s^2+a^2}-s)^k$, $(k>0)$	$\frac{ka^k}{t}J_k(at)$
$\frac{(\sqrt{s^2-a^2}+s)^\nu}{\sqrt{s^2-a^2}}$, $(\nu>-1)$	$a^\nu I_\nu(at)$
$\frac{1}{(s^2-a^2)^k}$, $(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}(\frac{t}{2a})^{k-(1/2)}I_{k-(1/2}(at)$
$\frac{1}{s\sqrt{s+1}}$	$\mathrm{erf}(\sqrt{t})$ $(\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\tau^2}\mathrm{d}\tau$=1- erfc (x))
$\frac{1}{s+\sqrt{s^2+a^2}}$	$\frac{J_1(at)}{at}$
$\frac{1}{(s+\sqrt{s^2+a^2})^N}$, N positive integers	$\frac{NJ_N(at)}{a^N t}$
$\frac{1}{\sqrt{s^2+a^2}(s+\sqrt{s^2+a^2})}$	$\frac{J_1(at)}{a}$
$\frac{1}{\sqrt{s^2+a^2}(s+\sqrt{s^2+a^2})^N}$	$\frac{J_N(at)}{a^N}$

Table 1: A List of Laplace and Inverse Laplace Transforms Related to Fractional Order Calculus

$F(s)$	$f(t)$
$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\|\sin kt\|$
$\frac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$
$\frac{1}{\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$
$\frac{1}{\sqrt{s}}e^{k/s}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$
$\frac{1}{s\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$
$\frac{1}{s\sqrt{s}}e^{k/s}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$
$\frac{1}{s^\nu}e^{-k/s}$, $(\nu > 0)$	$(\frac{t}{k})^{(\nu-1)/2}J_{\nu-1}(2\sqrt{kt})$
$\frac{1}{s^\nu}e^{k/s}$, $(\nu > 0)$	$(\frac{t}{k})^{(\nu-1)/2}I_{\nu-1}(2\sqrt{kt})$
$e^{-k\sqrt{s}}$, $(k > 0)$	$\frac{k}{2\sqrt{\pi t^3}}e^{-\frac{k^2}{4t}}$
$\frac{1}{s}e^{-k\sqrt{s}}$, $(k \geq 0)$	$\mathrm{erfc}(\frac{k}{2\sqrt{t}})$
$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$, $(k \geq 0)$	$\frac{1}{\sqrt{\pi t}}e^{-\frac{k^2}{4t}}$
$\frac{1}{s\sqrt{s}}e^{-k\sqrt{s}}$, $(k \geq 0)$	$2\sqrt{\frac{t}{\pi}}e^{-\frac{k^2}{4t}} - k\mathrm{erfc}(\frac{k}{2\sqrt{t}})$
$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})}$, $(k \geq 0)$	$-e^{ak}e^{a^2t}\mathrm{erfc}(a\sqrt{t}+\frac{k}{2\sqrt{t}})+\mathrm{erfc}(\frac{k}{2\sqrt{t}})$
$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}$, $(k \geq 0)$	$e^{ak}e^{a^2t}\mathrm{erfc}(a\sqrt{t}+\frac{k}{2\sqrt{t}})$
$\log\frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt}-e^{at})$
$\log\frac{s^2+a^2}{s^2}$	$\frac{2}{t}(1-\cos at)$
$\log\frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1-\cosh at)$
$\arctan\frac{k}{s}$	$\frac{1}{t}\sin kt$

Table 2: A List of Laplace and Inverse Laplace Transforms Related to Fractional Order Calculus (Continued)

$F(s)$	$f(t)$
$\frac{s^{\alpha-1}}{s^\alpha\mp\lambda}$, $\Re(s) > \|\lambda\|^{1/\alpha}$	$E_{\alpha,1}(\pm\lambda t^\alpha)$
$\frac{k!s^{\alpha-\beta}}{(s^\alpha\mp\lambda)^{k+1}}$, $\Re(s) > \|\lambda\|^{1/\alpha}$	$\varepsilon_k(t,\pm\lambda;\alpha,\beta)$
$\frac{k!}{(\sqrt{s}\mp\lambda)^{k+1}}$, $\Re(s) > \lambda^2$	$t^{\frac{k-1}{2}}E^{(k)}_{\frac{1}{2},\frac{1}{2}}(\pm\lambda\sqrt{t})$
$\frac{1}{s^\alpha}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$

Table 3: A List of Laplace and Inverse Laplace Transforms Related to Fractional Order Calculus (Mittag-Leffler function type)